About a resolvent formula

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Abstract

A resolvent formula, originally presented by Karner in his habilitation [3], is discussed. First the formula is considered abstractly and then it is demonstrated on an explicit example – the so called simplified Fermi accelerator.

1 Introduction

In his habilitation [3] Karner demonstrated on two explicit examples a resolvent formula adjusted to the case when the underlying Hilbert space was written as a tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ and the self-adjoint operator in question was of the form $A_1 \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes A_2$ plus a perturbation generally mixing the two factors $\mathcal{H}_1$ and $\mathcal{H}_2$. Usually the spectral decompositions of A_1 and A_2 are well known. In [3] Karner calls the formula a modified Krein formula. This is not quite exact as the formula depends in fact only on the described particular algebraic structure. Nevertheless it can be combined effectively with the Krein formula. Furthermore, some applications to spectral analysis have been proposed in [3] however this program doesn't seem to be completed entirely. Moreover the formula itself has to be extracted from the text of the habilitation.

Nevertheless we believe that the Karner's formula deserves a more detailed treatment. One reason for it is that the particular form of the considered operator, as mentioned above, occurs in various interesting situations. As a prominent example we may mention Floquet Hamiltonians introduced to study time dependent quantum systems [1, 5]. Thus in our short contribution we start from revealing the algebraic structure of the Karner's formula. To this end we treat it abstractly and confine ourselves for a while to a finite-dimensional case for then all the terms occurring in the formula are well defined. However this is not so obvious in some concrete examples when the Hilbert space is infinite-dimensional. It seems that one

has to consider each case separately in order to verify that the formula actually makes good sense. Here we discuss as an example the so called simplified Fermi accelerator [4].

2 Karner's formula

Proposition. *Suppose that $\mathcal{T}$ and $\mathcal{H}$ are two finite-dimensional vector spaces, and set $\mathcal{K} := \mathcal{T} \otimes \mathcal{H}$. Furthermore, let $\{Q_k\}_{k=1}^M$, $\{P_j\}_{j=1}^N$ be two complete sets of projectors on the vector space $\mathcal{T}$, i.e., $Q_k Q_{k'} = \delta_{kk'} Q_k$, $\sum_k Q_k = \mathbb{I}_{\mathcal{T}}$, $P_j P_{j'} = \delta_{jj'} P_j$, $\sum_j P_j = \mathbb{I}_{\mathcal{T}}$. To a set $\{\lambda_k\}_{k=1}^M$ of complex numbers and to a set $\{H_i\}_{i=0}^N$ of operators on $\mathcal{H}$ we relate the operators*

$$D := \textstyle\sum_{k=1}^M \lambda_k Q_k, \tag{1}$$

$$K_0 := D \otimes \mathbb{I}_{\mathcal{H}} + \mathbb{I}_{\mathcal{T}} \otimes H_0, \tag{2}$$

$$K := D \otimes \mathbb{I}_{\mathcal{H}} + \textstyle\sum_{j=1}^N P_j \otimes H_j, \tag{3}$$

$$\Lambda(z) := \textstyle\sum_{j=1}^N \sum_{k=1}^M P_j Q_k \otimes \Big((H_j + \lambda_k - z)^{-1} - (H_0 + \lambda_k - z)^{-1} \Big). \tag{4}$$

Then it holds

$$(K - z)^{-1} = \Big((K_0 - z)^{-1} + \Lambda(z) \Big) \Big(\mathbb{I} + [D \otimes \mathbb{I}_{\mathcal{H}},\, \Lambda(z)] \Big)^{-1}. \tag{5}$$

Remark. $(K-z)^{-1}$, $(K_0 - z)^{-1}$ and $\Lambda(z)$ are meromorphic functions with values in the space $\mathrm{Lin}(\mathcal{T} \otimes \mathcal{H})$, and all of them converge to 0 as $|z| \to +\infty$. Consequently $\det\Big(\mathbb{I} + [D \otimes \mathbb{I}, \Lambda(z)] \Big)$ is a meromorphic function as well, with a finite number of poles, and converging to 1 as $|z| \to +\infty$. This function has necessarily a finite number of zeroes and the equality (5) makes sense except of a finite number of points $z \in \mathbb{C}$.

Proof. Let us assume that $z \in \mathbb{C}$ is chosen so that all terms in (5) are well defined. Multiplying the relation (5) from the right by the expression $\Big(I + [D \otimes I, \Lambda(z)] \Big)(K_0 - z)$, and from the left by the expression $(K - z)$ one arrives at an equivalent identity, namely

$$K_0 - K = \left(\Lambda(z)(D \otimes I) + \Big(\sum_{j=1}^N P_j \otimes (H_j - z) \Big) \Lambda(z) \right) (K_0 - z)\,. \tag{6}$$

Using the equalities

$$\Lambda(z)(D \otimes \mathbb{I}) = \sum_j \sum_k P_j Q_k \otimes \lambda_k \Big((H_j + \lambda_k - z)^{-1} - (H_0 + \lambda_k - z)^{-1} \Big),$$

$$\Big(\sum_{j=1}^{N} P_j \otimes (H_j - z)\Big)\Lambda(z)$$
$$= \sum_j \sum_k P_j Q_k \otimes (H_j - z)\Big((H_j + \lambda_k - z)^{-1} - (H_0 + \lambda_k - z)^{-1}\Big),$$

one can show that the RHS of (6) equals

$$\begin{aligned}
&\left(\sum_j \sum_k P_j Q_k \otimes (H_j + \lambda_k - z)\Big((H_j + \lambda_k - z)^{-1} - (H_0 + \lambda_k - z)^{-1}\Big)\right)\\
&\qquad \times (K_0 - z)\\
&= (K_0 - z) - \Big(\sum_j \sum_k P_j Q_k \otimes (H_j + \lambda_k - z)(H_0 + \lambda_k - z)^{-1}\Big)(K_0 - z)\\
&= -\Big(\sum_j \sum_k P_j Q_k \otimes (H_j - H_0)(H_0 + \lambda_k - z)^{-1}\Big)(K_0 - z)\\
&= -\Big(\sum_j P_j \otimes (H_j - H_0)\Big)\Big(\sum_k Q_k \otimes (H_0 + \lambda_k - z)^{-1}\Big)(K_0 - z)\\
&= (K_0 - K)\Big(\sum_k Q_k \otimes (H_0 + \lambda_k - z)\Big)^{-1}(K_0 - z) = K_0 - K\,,
\end{aligned}$$

and this completes the verification. ∎

3 Example: simplified Fermi accelerator

We set $\mathcal{T} = L^2([\,0,T\,], dt)$, $\mathcal{H} = L^2([\,0,1\,], dx)$, and so $\mathcal{K} = L^2([\,0,T\,]\times[\,0,1\,], dt\,dx)$. We identify $\mathcal{K}$, as usual, with $L^2([\,0,T\,], \mathcal{H}, dt)$. We set further $D = -i\partial_t$ with periodic boundary conditions, and $H_0 = -\partial_x^2$ with Neumann boundary conditions. Both D and H_0 are self-adjoint operators with discrete spectra. The diagonalization (1) of D is now replaced by an infinite sum, with $\lambda_k = k\omega$, $k \in \mathbb{Z}$, where $\omega := 2\pi/T$, and Q_k's are the orthogonal projectors on the eigen-functions $\chi_k(t) := T^{-1/2}\exp(i\,k\omega t)$. Clearly, the operator $K_0 = -i\partial_t \otimes \mathbb{I} - \mathbb{I} \otimes \partial_x^2$ is self-adjoint with a pure point spectrum.

Let us now make a small digression and consider a perturbation H_g of H_0 written in the form sense as $H_g := H_0 + g\,\tau^*\tau$ where $\tau : \mathcal{H} \to \mathbb{C}$ is the trace operator: $\tau u := u(0)$. Of course, this means nothing but that, in the case of H_g, the boundary condition at the point $x = 0$ reads $(\partial_x f)(0) = g\,f(0)$ while the boundary condition at the point $x = 1$ is still of Neumann type. In fact, H_g is an entire analytic family of type B [2, ch.VII §4] in the variable g.

For a later convenience let us also examine the resolvents of H_0 and H_g. We set $R_0(z) := (H_0 - z)^{-1}$ and $R_g(z) := (H_g - z)^{-1}$. It is easy to calculate the Green

function corresponding to H_0 explicitly, namely

$$G_0(x,y) = -\frac{\cos(\sqrt{z}\,x)\cos\left(\sqrt{z}(y-1)\right)\vartheta(y-x) + \{x \leftrightarrow y\}}{\sqrt{z}\,\sin(\sqrt{z})} \tag{7}$$

where $\vartheta(x)$ is the Heaviside step function. Particularly,

$$\tau R_0(z)\tau^* = -\frac{\cot(\sqrt{z})}{\sqrt{z}}\,. \tag{8}$$

The two resolvents are related by the equality $R(z) = R_0(z) - gR(z)\tau^\star\tau R_0(z)$, and so

$$R_g(z) - R_0(z) = \frac{-g}{1 + g\,\tau R_0(z)\tau^*}\,R_0(z)\tau^*\tau R_0(z)\,. \tag{9}$$

Here $R_0(z)\tau^*\tau R_0(z)$ is a rank-one operator with the norm

$$\begin{aligned} \|R_0(z)\tau^*\tau R_0(z)\|^2 &= \tau R_0(\bar{z})R_0(z)\tau^*\,\|R_0(\bar{z})\tau^*\tau R_0(z)\| \\ &= (\tau R_0(\bar{z})R_0(z)\tau^*)\,(\tau R_0(z)R_0(\bar{z})\tau^*) \\ &= \left(\frac{\operatorname{Im}\tau R_0(z)\tau^*}{\operatorname{Im} z}\right)^2. \end{aligned} \tag{10}$$

Suppose now that $g(t)$ is a T-periodic real function. Following [4] we call the time-dependent quantum system determined by the Hamiltonian $H(t) \equiv H_{g(t)}$ a simplified Fermi accelerator. The corresponding Floquet Hamiltonian $K := -i\partial_t + H(t)$ has the same structure as given in (3) provided one replaces the sum $\sum_j P_j \otimes H_j$ by the direct integral

$$\int_0^{T\oplus} H(t)\,dt\,. \tag{11}$$

This means that the family of projectors $\{P_j\}_j$ is formally substituted by the spectral decomposition of the multiplication operator $X \in \mathcal{B}(\mathcal{T})$, $(Xf)(t) := t\,f(t)$, which has, however, an absolutely continuous spectrum. Proceeding this way one finds that the definition (4) of the operator $\Lambda(z)$ has to be modified as follows:

$$\Big(\Lambda(z)\psi\Big)(t) := \sum_{k=-\infty}^{+\infty} \chi_k(t)\Big(R_{g(t)}(z-k\omega) - R_0(z-k\omega)\Big)\int_0^T \overline{\chi_k(\tau)}\,\psi(\tau)\,d\tau \tag{12}$$

where $\psi \in L^2([\,0,T\,],\mathcal{H},dt)$ and $z \in \mathbb{C}\setminus\mathbb{R}$.

The definition (12) implies that

$$\Big(\Lambda(z)\,\chi_k\otimes f\Big)(t) = \chi_k(t)\Big(R_{g(t)}(z-k\omega) - R_0(z-k\omega)\Big)f, \quad \forall k \in \mathbb{Z} \text{ and } f \in \mathcal{H}, \tag{13}$$

where we have used the convention $(u\otimes f)(t) := u(t)\,f$, with $u \in \mathcal{T}$ and $f \in \mathcal{H}$. So $\Lambda(z)$ is densely defined. To show that $\Lambda(z)$ is even bounded we shall treat the

RHS of (12) perturbatively assuming that z is separated from the real axis, i.e., $|\mathrm{Im}\, z| \geq s_0 > 0$, and $g \in C^0$, with the supremum norm $\|g\|_\infty$ being sufficiently small with respect to s_0. Relying on the estimates

$$|\tau R_0(z)\tau^*| = \left|\frac{\cot(\sqrt{z})}{\sqrt{z}}\right| \leq \frac{2}{|\mathrm{Im}\, z|}\sqrt{1+\frac{|\mathrm{Im}\, z|}{4}} \leq \frac{2}{s_0}\sqrt{1+\frac{s_0}{4}} =: \alpha(s_0) \tag{14}$$

and

$$\|R_0(z)\tau^*\tau R_0(z)\| \leq \left|\frac{\tau R_0(z)\tau^*}{\mathrm{Im}\, z}\right| \leq \frac{\alpha(s_0)}{s_0}\,, \tag{15}$$

we find that if $\|g\|_\infty < 1/\alpha(s_0)$ then

$$\begin{aligned} \Big(\Lambda(z)\psi\Big)(t) \;\; = \;\; & \sum_{n=0}^{\infty}(-1)^{n+1}\, g(t)^{n+1} \sum_{k=-\infty}^{+\infty} \chi_k(t) \left(\tau R_0(z-k\omega)\tau^*\right)^n \\ & \times R_0(z-k\omega)\tau^*\tau R_0(z-k\omega) \int_0^T \overline{\chi_k(\tau)}\, \psi(\tau)\, d\tau \end{aligned} \tag{16}$$

and so

$$\begin{aligned} \|\Lambda(z)\psi\| \;\; \leq \;\; & \sum_{n=0}^{\infty} \|g\|_\infty^{n+1} \Bigg(\sum_{k=-\infty}^{+\infty} |\tau R_0(z-k\omega)\tau^*|^{2n} \\ & \times \|R_0(z-k\omega)\tau^*\tau R_0(z-k\omega)\|^2 \left\| \int_0^T \overline{\chi_k(\tau)}\, \psi(\tau)\, d\tau \right\|_{\mathcal{H}}^2 \Bigg)^{1/2} \\ \leq \;\; & \frac{\|g\|_\infty\, \alpha(s_0)}{s_0\big(1-\|g\|_\infty\, \alpha(s_0)\big)}\, \|\psi\|\,. \end{aligned} \tag{17}$$

Next let us consider the commutator $[\, D\otimes\mathbb{I},\, \Lambda(z)\,]$. One deduces from (13) immediately that

$$\Big([\, D\otimes\mathbb{I},\, \Lambda(z)\,]\, \chi_k \otimes f\Big)(t) = -i\, \chi_k(t)\, \partial_t \Big(R_{g(t)}(z-k\omega) - R_0(z-k\omega)\Big) f\,. \tag{18}$$

Assuming that $g \in C^1$ we get a relation similar to (16), namely

$$\begin{aligned} \Big([\, D\otimes\mathbb{I},\, \Lambda(z)\,]\psi\Big)(t) \;\; = \;\; & -i\, g'(t) \sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\, g(t)^n \sum_{k=-\infty}^{+\infty} \chi_k(t) \\ & \times \left(\tau R_0(z-k\omega)\tau^*\right)^n R_0(z-k\omega)\tau^*\tau R_0(z-k\omega) \\ & \times \int_0^T \overline{\chi_k(\tau)}\, \psi(\tau)\, d\tau\,, \end{aligned} \tag{19}$$

and consequently the estimate

$$\big\|[\, D\otimes\mathbb{I},\, \Lambda(z)\,]\psi\big\| \leq \frac{\|g'\|_\infty\, \alpha(s_0)}{s_0\big(1-\|g\|_\infty\, \alpha(s_0)\big)^2}\, \|\psi\|\,. \tag{20}$$

Since, apart of the problems with the precise definition of the operator $\Lambda(z)$, the algebraic structure remains the same as in the finite-dimensional case we conclude that the formula (5) extends, as it is, also to our example of simplified Fermi accelerator provided $|\mathrm{Im}\, z| \geq s_0$, $\|g\|_\infty\, \alpha(s_0) < 1$ and

$$\frac{\|g'\|_\infty\, \alpha(s_0)}{s_0\big(1 - \|g\|_\infty\, \alpha(s_0)\big)^2} < 1\,. \tag{21}$$

Of course, from the point of view of spectral analysis, a truly interesting result would involve the limit $|\mathrm{Im}\, z| \downarrow 0$. There is no doubt that this goal requires a much more subtle analysis than that based on the elementary estimate (14), and this program goes well beyond the scope of this short contribution. We hope anyway to have shown that the formula (5) may be given a good sense.

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